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Geometrization of Magnetohydrodynamic Equations via Lie Groups

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The equations of magnetohydrodynamics of a perfect fluid are classified with respect to the Coriolis parameter, and all essentially different solutions of rank one are indicated. The geometry of streamlines is discussed.

1. INTRODUCTION

Transformation group methods have been extensively used for analyzing and classifying differential equations. For the magnetohydrodynamic equations with which we are concerned here, it may be remarked that the classical procedures require considerable modifications to yield acceptable results. One of the initial attempts in this direction seems to have been that of Kucharczyk (1964), following an earlier model of Yano (1955). Ladikov (1962) studied the geometry of streamlines. Recently Singh and Choubey (1985) and Singh and Tripathi (1986) obtained the geometry of streamlines in the case of magnetogas flows with magnetic field lines acting along a fixed direction and under the influence of a Coriolis force, and considered the physical application of MHD equations. In the present paper the problem of the group classification of the system (6) is solved, an optimum system of one-parameter subgroups is determined, and all essentially different solutions are indicated. The method followed in this paper seems to have a physical advantage over the available classical techniques of flow classification.

2. FLOW EQUATIONS

Under the influence of a rotating reference frame the magnetohydrodynamic equations of a perfect fluid are

$$\operatorname{div} \mathbf{v} = 0 \tag{1}$$

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$$(\mathbf{v}\cdot \text{grad})\mathbf{v}+2\boldsymbol{\omega}\mathbf{x}\mathbf{v}=-\text{grad}\, p+\mu \text{ curl }\mathbf{H}\times\mathbf{H} \tag{2}$$

$$\text{curl }\mathbf{vxH}=\mathbf{0} \tag{3}$$

$$\text{div }\mathbf{H}=0 \tag{4}$$

where **v**, **H** and **ω** are velocity, magnetic field, and angular velocity vectors, respectively, and μ is the magnetic permeability. Equation (4) asserts an additional condition on **H**, expressing the absence of magnetic poles in the flow. Following Surayanarayana (1965), we take the plane transverse flow as governed by

$$\begin{aligned} &\frac{\partial u}{\partial x}+\frac{\partial v}{y}=0\\ &u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\omega^* v=-\frac{\partial p^*}{\partial x}\\ &u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\omega^* u=-\frac{\partial p^*}{\partial y}\end{aligned} \tag{5}$$

where $\mathbf{v}=(u, v)$, $p^*=(p+\frac{1}{2}\mu H^2)$ is the total pressure, and $\omega^*=2\omega$ is the Coriolis parameter. The factors $\omega^* u$ and $\omega^* v$ represent components of acceleration produced by the Coriolis force due to the rotation of the earth, and the parameter $\omega^*(y)$ can be an arbitrary function of y. For an arbitrary $\omega^*(y)$, the system (5) admits a certain group of transformations G. The special forms of the function $\omega^*(y)$ for which the fundamental group admitted by the system (5) is wider than G are to be determined.

3. CLASSIFICATION OF EQUATIONS

3.1. For any arbitrary function $\omega^*(y)$ the basic operators of the related Lie algebra are of the form

$$X_1=\partial/\partial p, \qquad X_2=\partial/\partial x \tag{6}$$

For other forms of $\omega^*(y)$ we find the following results.

3.2. If $\omega^*=y^{m-1}(m\neq 1)$, the operator

$$X_3^1=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+mu\frac{\partial}{\partial u}+2mp\frac{\partial}{\partial p} \tag{7}$$

is added to the operators (5).

3.3. If $\omega^*=e^{my}(m\neq 0)$, the operator

$$X_3^2=\frac{\partial}{\partial y}+mu\frac{\partial}{\partial u}+mv\frac{\partial}{\partial v}+2mp\frac{\partial}{\partial p} \tag{8}$$

is added to (5).

3.4. If $\omega^*=1$, the fundamental group is generated by five operators, with the three operators

$$X_3^3 = x\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+2p\frac{\partial}{\partial p}$$
$$X_4=\frac{\partial}{\partial y}, \qquad X_5=-y\frac{d}{dx}+x\frac{\partial}{\partial y}-x\frac{\partial}{\partial y}-v\frac{\partial}{\partial u}+u\frac{\partial}{\partial v} \tag{9}$$

added to (5).

3.5. If $\omega^*=0$, the group is generated by six operators, with the operators

$$X_3^4= \frac{\partial}{\partial x}+y\frac{\partial}{\partial y}, \qquad X_4=\frac{\partial}{\partial y}, \qquad X_5=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}-v\frac{\partial}{\partial u}+u\frac{\partial}{\partial v}$$
$$X_6=\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+2p\frac{\partial}{\partial p} \tag{10}$$

added to (5).

4. SOLUTION FOR $\omega^*=$ CONST

Let us construct the essentially different solutions for the first three cases.

4.1. If $\omega^*(y)$ is an arbitrary function, the optimum system is generated by operators (5). The subgroup with operator X_1 is eliminated, since for it the necessary condition of the existence of an invariant solution is not satisfied, while the subgroup with operator X_2 provides a solution that depends on y. The substitution of these into (5) yields the system

$$vu'=\omega^*v=0, \qquad vv'+\omega^*u=-p', \qquad v'=0$$

where the prime denotes differentiation. From the last equation we have $v=v_0$; for $v_0\neq 0$ the integration yields

$$u=u_0+\lambda(y), \qquad p=p_0-\tfrac{1}{2}\lambda^2+u_0$$

$\lambda'(y)=\omega^*(y)$ and f is an arbitrary function of its argument. If, however, $v_0=0$, then

$$u=u_0+f(y), \qquad p=p_0^+\int \omega^*(y)[u_0+f(y)]\,dy$$

In the first case the streamlines are defined by the formula

$$u_0y-v_0x+\int (y)\,dy=\text{const}$$

and in the second the streamlines are *straight lines* parallel to the x axis.

4.2. For $\omega^* = y^{m-1}$ the optimum system is generated by operators X_1, X_3^1 and $X_2 + aX_1$, and the solution for the subgroup with operator X_3^1 is of the form

$$u = x^m U(\xi), \qquad v = x^m V(\xi), \qquad p = x^{2m}(\xi), \qquad \xi = y/x$$

The last subgroup yields a solution of the form

$$u = U(y), \qquad v = V(y), \qquad p = ax + P(y)$$

Substituting in (5) and integrating, we find the following solutions:

$$u = u_0 + \frac{1}{m} y^m - \frac{\alpha}{v_0} y, \qquad v = v_0$$

$$p = p_0 + ax - \frac{1}{2m^2} y^{2m} + \frac{\alpha}{v_0(m+1)} y^{m+1} - \frac{u_0}{m} y^m$$

It is assumed that in this case $m \neq 0$, $m \neq -1$.

In meteorological problems the Coriolis parameter is often approximately by a linear function, which corresponds to $m = 2$. In this case streamlines are *cubic parabolas*, and the solution may be treated as defining a *crest type of flow*. Meteorological observations show that such flows result in the formation of fronts with an abrupt change of weather.

If $m = 0$, the solution is of the form

$$u = u_0 + \ln y - \frac{\alpha}{v_0} y, \qquad v = v_0$$

$$p = p_0 + ax - \tfrac{1}{2} \ln^2 y + \frac{\alpha}{v_0} y - u_0 \ln y$$

For $m = -1$

$$u = u_0 - \frac{1}{y} - \frac{\alpha}{v_0} y, \qquad v = v_0$$

$$p = p_0 + ax - \frac{1}{2y^2} + \frac{\alpha}{v_0} \ln y + \frac{u_0}{y}$$

4.3. For $\omega^* = e^{my}$ the optimum system of one parametric subgroup is generated by operators $aX_1 + X_2$ and $X_3^2 + aX_2$. The first subgroup yields solutions of the form

$$u = U(y), \qquad v = V(y), \qquad p = ax + P(y)$$

Putting these into system (5) and integrating, we obtain

$$u = u_0 + \frac{1}{m} e^{my} - \frac{\alpha}{v_0} y v = v_0$$

$$p = p_0 + ax + \frac{1}{m^2}\left[\frac{\alpha}{v_0}(my-1) - \tfrac{1}{2} - mu_0\right] e^{my}$$

which is similar to that for *crest type of flow.*

The solution for the second subgroup is of the form

$$u = e^{my}U(\xi), \qquad v = e^{my}V(\xi), \qquad p = e^{2my}P(\xi), \qquad \xi = y - ax$$

and the unknown functions satisfy the following system of equations:

$$UU' + V'(mU - \alpha U') - V = -P'$$

$$UV' + V(mV - \alpha V') + U = -2mP + \alpha P'$$

$$U' + mV - \alpha V' = 0$$

Another solution of this system is

$$U = a + v_0\, e^{k\xi}, \qquad V = v_0\, e^{k\xi}, \qquad P = bv_0\, e^{k\xi} - a/2m$$

$$a = \frac{(\alpha-1)^2}{m(\alpha^2 - 2\alpha - 1)}, \qquad b = \frac{1-\alpha^2}{m(\alpha^2 - 2\alpha - 1)}, \qquad k = \frac{m}{\alpha - 1}$$

The streamlines $\psi = \text{const}$ are specified by

$$x = y - [(\alpha - 1)/m]\ln(\text{const} - a/m\, e^{my})$$

and represents a curve tending to a *straight line.* This solution can simulate the flow in a cumulative stream.

5. SOLUTION FOR CONSTANT CORIOLIS PARAMETER

Before proceeding with the construction of the solution for $\omega^* = 1$ and $\omega^* = 0$, we note that the input system (5) for a constant Coriolis parameter can be replaced by an equivalent system by substituting the vorticity $\Omega = u_y - v_x$ for one of the unknown functions. For $\Omega = 0$ system (5) is equivalent to the Cauchy-Riemann equations $u_y - v_x = 0$ and $u_x - v_y = 0$, which admit an infinite group of transformations. Because of this we subsequently seek only solutions of nonzero vorticity. It will be necessary to use the input equations in polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad u = \Omega\cos\phi, \qquad v = \Omega\sin\phi$$

The expressions for the vorticity and the stream functions are

$$\Omega = \cos(\phi - \theta)\left(\frac{1}{r}\Omega_\phi - {}^{\Omega}\phi_r\right) - \sin(\phi - \theta)\left(\Omega_r + \frac{\Omega}{r}\phi_0\right)$$

$$\psi_r = -\sin(\phi - \theta), \qquad r\psi_0^{-1} = \cos(\phi - \theta) \tag{11}$$

The arbitrary constants of integration of equations in polar coordinates are denoted by Ω_0, p_0, and ϕ_0.

5.1. Solutions for $\omega^* = 1$. The optimum system is generated by seven operators,

$$X_1, \quad X_2, \quad X_3^3, \quad X_1 + X_2, \quad X_2 + X_5, \quad X_3^3 + aX_5$$

Let us write (5) in polar coordinates. Taking into account trigonometric identities, we obtain the following system:

$$\frac{\Omega^2}{r}\phi_0 + \Omega\sin(\phi-\theta) = p_r$$

$$\Omega^2\theta_r + \Omega\cos(\phi-\theta) = \frac{1}{r}p_\theta \tag{12}$$

$$\cos(\phi-\theta)\left(\Omega_r + \frac{\Omega}{r}\phi_0\right) - \sin(\phi-\theta)\left(\Omega\phi_r - \frac{1}{r}\Omega_\theta\right) = 0$$

In the same variables the operators X_3^3, X_5 assume the form

$$X_3^3 = r\frac{\partial}{\partial r} + \frac{\partial}{\partial\Omega} + 2P\frac{\partial}{\partial p}, \qquad X_5 = \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}$$

5.1.1. The subgroup with operator X_1 does not yield invariant solutions, since the necessary condition for the existence of such solutions is not satisfied.

5.1.2. The subgroup with operator X_2 yields the solutions, which depend only on y. Integrating (5), we obtain

$$u = u_0 + y, \qquad v = v_0, \qquad p = p_0 - \tfrac{1}{2}y^2 - u_0 y \qquad \text{if} \quad v_0 \neq 0$$

$$u = -f'(y), \qquad v = 0, \qquad p = f(y) \qquad \text{if} \quad v_0 = 0$$

In the first case the vortex $\Omega = 1$ and the streamlines are represented by a *set* of Parabolas. In second case $\Omega = -f''$ and the streamlines are *straight lines parallel to the x axis.*

5.1.3. For the subgroup with operator X_3^3, we seek the solution in the form $\Omega = r\omega(\theta)$, $\phi = \Phi(\theta)$, and $p = r^2 P(\theta)$; substituting into (12), we obtain the equations

$$\omega^2\phi' + \omega\sin(\phi-\theta) = 2P, \qquad \omega\cos(\phi-\theta) = -P'$$

$$\cos(\phi-\theta)\,\omega(1+\phi') + \sin(\phi-\theta)\,\omega' = 0$$

In integrating this system, we consider two cases: $\phi' = 0$ and $\phi' \neq 0$. For $\phi' = 0$ the solution is of the form

$$\Omega = \Omega_0 r\sin(\theta-\phi_0), \qquad \phi = \phi_0, \qquad p = -\tfrac{1}{2}\Omega_0 r^2\sin^2(\theta-\phi_0)$$

It can be shown that $\Omega = \Omega_0$. This solution yields streamlines that are *parallel straight lines at an angle ϕ to the X_0 axis.* Since the diirection of coordinate axis was not specified, the x axis can be made to coincide with that of the velocity vector, i.e., we can set $\phi_0 = 0$. The absolute value of the velocity vector is proportional to y.

The second case, in which $\phi_0 \neq 0$, yields a solution that can be written in the parametric form

$$\Omega = r\left(\frac{p_0}{z+1}\right)^{1/2}, \qquad \phi = \theta + \sin^{-1}\left[\Omega_0 \frac{z+2}{(z+1)^{1/2}}\right]$$

$$p = r^2\left[p_0 + \frac{\Omega_0(z+1)}{z-1}\left(\frac{p_0}{2}\right)^{1/2}\right]$$

$$\phi = \phi_0 + \tfrac{1}{2}\sin^{-1}\frac{(1-2\Omega^2)(z+1) - 2\Omega_0^2}{(z+1)[(1-4\Omega^2)]^{1/2}}$$

For the vortex and stream functions we have

$$\Omega = -\left(\frac{2p_0}{\Omega_0}\right)^{1/2}$$

$$\psi = \psi_0 - \frac{\Omega_0}{2} r^2 \frac{z+2}{z+1} = -\Omega_0 + ar^2 + br^2 \sin(2\theta - 2p_0)$$

$$a = \frac{1}{2\Omega_0}\left(\frac{p_0}{2}\right)^{1/2}, \qquad b = \left[\left(\frac{p_0}{2}\right)\left(\frac{1}{4\Omega_0} - 1\right)\right]^{1/2}$$

By a suitable selection of coordinate axes it is possible to obtain $\phi_0 = 0$ and the streamlines are defined by

$$a(x^2 + y^2) + 2bxy + \text{const} = 0$$

i.e., they represent a set of ellipses in the system of coordinate turned by 45°.

5.1.4. The subgroup with operator X_5 yields a solution of the form $\Omega = \omega(r)$, $\phi = \phi(r)$, and $p = \rho(r)$. Substituting into (12), we obtain the system

$$\frac{1}{r}\omega^2 + \omega \sin\phi = P', \qquad \omega^2\phi + \omega \cos\phi' = 0$$
$$\cos\phi\,[\omega' + (1/r)\omega] - \sin\Phi\omega\Phi' = 0 \qquad (13)$$

The last equation yields the first integral $r\omega\cos\Phi = \Omega_0$. We can now write the general solution

$$\Omega = \frac{\Omega_0}{r}\left[1 + \left(\phi_0 - \frac{r^2}{2\Omega_0}\right)^2\right]^{1/2}$$

$$\phi = \theta + \mathrm{tg}^{-1}\left(\phi_0 - \frac{r^2}{2\Omega_0}\right)$$

$$p = p_0 - \frac{r^2}{8} - (1 + \phi_0^2)\frac{\Omega_0^2}{2r^2}$$

The streamlines

$$\theta = \phi_0 \ln - \tfrac{1}{4}r^2/\Omega_0 + \text{const}$$

are represented by *set of spiral lines* with the source ($\Omega_0 > 0$) or sink ($\Omega_0 < 0$) at the coordinate origin.

For $\Omega_0\phi_0 < 0$ the angle θ along the streamline varies monotonically, while for $\Omega_0\phi_0 > 0$ the monotonicity breaks down. It can be shown by direct calcuation that for such flow the vorticity is constant and equal to unity.

5.1.5. For the subgroup with operator $X_1 + X_2$ the solution is sought in the form $u = U(y)$, $v = V(y)$, and $p = x + \rho(y)$. Putting these into (5), we get

$$u = u_0 + \frac{v_0 - 1}{v_0} yv = v_0$$

$$p = p_0 + x + u_0 y - \frac{v_0 - 1}{2v_0} y^2, \qquad \Omega = \frac{v_0 - 1}{2v_0}$$

If $v_0 = 1$, the vortex is nonzero and the streamlines are *parabolas* with their axis parallel to the x axis. If $v_0 = 0$, the system (5) becomes inconsistent.

5.1.6. For the subgroup with operator $X_1 + X_5$, the solution is of the form $\Omega = \omega(r)$, $\theta = \theta + \Phi(r)$, and $p = \theta + \rho(r)$.

The unknown function is determined by the system (5) in which $-1/r$ is substituted into the right-hand side of the second equation. Integration of the obtained equations yields the solution

$$\Omega = \frac{\Omega_0}{r}\left[1 + \left(\phi_0 - \frac{\Omega_0 + 1}{2\Omega_0^2} r^2\right)^2\right]^{1/2}$$

$$\phi = \theta + \mathrm{tg}^{-1}\left(\phi_0 - \frac{\Omega_0 + 1}{2\Omega_0^2} r^2\right)$$

$$p = p_0 + -\left(1 - \frac{1}{\Omega_0^2}\right)\frac{r^2}{8} - (1 + \phi_0^2)\frac{\Omega_0^2}{2r^2} - \phi_0 \ln r$$

The solution is similar to that derived in (5.1.4) and the vortex $\Omega = 1 + 1/\Omega_0$ is nonzero throughout.

5.1.7. The last subgroup with operator $X_3^3 + aX_5$ generates a solution of the form

$$\Omega = r\omega(\xi), \qquad \phi = \theta + \Phi(\xi), \qquad p = r^2 P(\xi)$$

were $\xi = r\,e^{-\theta}$.

5.2. For $\omega^* = 0$, (5) in polar coordinates is of the form

$$\frac{\Omega^2}{r}\,\phi_\theta = P_r, \qquad \Omega^2 \phi_r = -\frac{1}{r}\,P_\theta$$

$$\cos(\phi - \theta)\left(\Omega_r - \frac{\Omega}{r}\,\phi_\theta\right) - \sin(\phi - \theta)\left(\phi_r - \frac{1}{r}\,\Omega_0\right) = 0$$

In certain instances the last equation will be represented in a somewhat different form by the substitution for the derivatives of ϕ of their expressions in the first two equations. In polar coordinates the operators X_3^4 and X_6 are of the simpler form

$$X_3^4 = r\frac{\partial}{\partial r}, \qquad X_6 = \frac{\partial}{\partial \Omega} + 2p\frac{\partial}{\partial p}$$

The optimum system of one-parameter subgroups is generated by 11 operators, which for convenience are divided into two classes:

1. $X_2,\ X_3^4,\ X_5,\ X_3^4 + X_5$
2. $X_1 + X_2,\ X_1 + X_3^4,\ X_2 + X_6,\ X_1 + X_5,\ X_1 + X_3^5 + X_5,$
   $X_3^4 + X_6,\ X_6 + X_5$

Direct calculation shows that operators of the first class yield solutions that define vorticity-free flows only; hence, in accordance with what has been previously stated, we restrict the analysis to operators of the second class.

5.2.1. The subgroup with operator $X_1 + X_2$ yields a trivial solution $u = u_0 - y/v_0$, $v_0 = 0$, and $p = p_0$. The streamlines are *parabolas* with their axis parallel to the x axis. For $v_0 \neq 0$, (5) becomes inconsistent.

5.2.2. For the subgroup with operator $X_1 + X_3^4$ we seek a solution of the form

$$\Omega = \omega(\theta), \qquad \phi = \Phi'(\theta), \qquad p = \ln r + P(\theta)$$

Substituting in (12), we obtain

$$\omega^2 \Phi' = 1, \qquad \cos(\Phi - \theta)\,\omega\Phi + \sin(\Phi - \theta)\omega' = 0, \qquad P = p_0$$

We use the first of these equations to determine ω in terms of the reduce the second equation $F''=2(1+F')^2$ etg F by introducing the new function $F=\phi-\theta$ and substituting for ω. The order of this equation is reduced by one by setting $F'=ZF$. The lower order equation is integrated and yields the following dependence of F on Z:

$$\Omega_0 \sin^2 F=(Z+1)\exp\frac{1}{Z+1}$$

Let us consider Z as a parameter that reduces the solution to one quadrature. The solution itself can be represented in the parametric form

$$\begin{aligned}
\Omega &= t^{1/2}\\
\phi &= \theta+F(t)\\
p &= p_0+\ln r\\
F(t) &= \sin^{-1}\frac{e^{t/2}}{(\Omega_0 t)^{1/2}}\\
\theta &= \phi_0-\frac{1}{2}\int\frac{dt}{\Omega_0 t\, e^{-t}-1}\\
t &= \frac{1}{Z+1}
\end{aligned}$$

The vorticity $\Omega=-\Omega_0^{1/2}\, e^{-t/2}/r$ is nonzero throughout. The streamlines are represented by a set of *spiral lines* and the solution exists only for $\Omega_0>e$.

5.2.3. The subgroup X_2+X_6 generates a solution of the form $u=e^xU(y)$, $v=e^xV(y)$, and $p=e^{2x}P(y)$. The substitution of these expressions into (5) yields

$$VV''-V'^2=2P_0, \qquad P=P_0 \tag{14}$$

If $p_0=0$, then the solution has the form

$$u=-u_0v_0\, e^{x+v_0y}, \qquad v=v_0\, e^{x+v_0y}, \qquad p=0$$

and the streamlines are *straight lines* $x+v_0y=$ const. But $P_0\neq 0$, and the second equation (14), after a single integration, yields

$$V'^2=AV^2-B$$

where A and B are arbitrary constants. Several cases must be considered, depending on the signs of A and B.

(1) If $A=v_0^2$ and $B/v_0^2=-p_0^2$, the solution is of the form

$$\begin{aligned}
u &= -v_0p_0\, e^x\, \mathrm{ch}(v_0y+u_0)\\
v &= p_0\, e^x\, \mathrm{sh}(v_0y+u_0)\\
p &= -\tfrac{1}{2}v_0^2p_0^2\, e^{2x}
\end{aligned}$$

If $p_0 < 0$ and $v_0 > 0$, the streamlines approach the *straight line* $y = y_0 = u_0/v_0$. If however, $p_0 > 0$ and $v_0 < 0$, the streamlines move away from the straight line $y = y_0$. Other combinations of these inequalities yield a similar flow pattern, except that the velocity vector changes to the opposite direction.

(2) When $A = v_0^2$ and $B/v_0^2 = p_0^2$, the integration of (15) yields the solution

$$u = v_0 p_0 \, e^x \cos(v_0 y + u_0)$$
$$v = p_0 \, e^x \sin(v_0 y + u_0)$$
$$p = -\tfrac{1}{2} v_0^2 p_0^2 \, e^{2x}$$

The solution is periodic with respect to y. The flow is divided into bands π/v_0 wide, inside which the velocity vector monotonically changes to the opposite direction along the streamlines from one boundary of the band to the other. The flow resembles that with *contact discontinuity* of equations for a compressible fluid.

The vortex is defined by the formula

$$\Omega = p_0(v_0^2 - 1) \, e^x \sin(v_0 y + u_0)$$

For $|v_0| \neq 1$ the vortex is nonzero. For the first two cases it is nonzero for any values of the constants.

5.2.4. For the subgroups with operator $X_1 + X_5$ the solution is sought in the form $\Omega = \omega(r)$, $\phi = \theta + \Phi(r)$, and $p = \theta + P(r)$. Substituting into ( ) and integrating, we get

$$\Omega = \frac{\Omega_0}{r}\left[1 + \left(\phi_0 - \frac{r^2}{2\Omega_0^2}\right)^2\right]^{1/2}$$

$$\phi = \theta + \mathrm{tg}^{-1}\left(\phi_0 - \frac{r^2}{2\Omega_0^2}\right)$$

$$p = p_0 + \frac{1}{8\Omega_0} r^2 - (1 + \phi_0^2)\frac{\Omega_0^2}{2r^2} - \phi_0 \ln r$$

The constant Ω_0 is assumed to be nonzero, since otherwise the system of equations would be consistent. The vortex is determined by $\Omega = 1/\Omega_0$ and solution is similar to that derived in 5.1.4.

5.2.5. The solution generated by the subgroup with the operator $X_1 + X_3^4 + X_5$ can be represented by $\Omega = \omega(\xi)$, $\phi = \theta + \Phi(\xi)$, and $p = \theta + P(\xi)$, where $\xi = r\, e^{-\theta}$ is the new independent variable.

5.2.6. For the subgroup with the operator $X_3^4 + X_6$, the solution is specified by the formulas $\Omega = r\omega(\theta)$, $\phi = 0(\theta)$, and $p = r^2 P(\theta)$, where the unknown functions are determined by the equations

$$p = p_0, \qquad \omega^2 \Phi' = 2p_0, \qquad \cos(\Phi - \theta)\,(\omega^2 + 2p_0) + \sin(\Phi - \theta)\,\omega\omega' = 0$$

For $p_0 = 0$ the solution of the system is $\Omega_0 r \sin(\theta - \phi_0)$, $\phi = \phi_0$, and $p = 0$, which in the system of coordinates rotated by an angle ϕ_0 corresponds to the *couette type flow*. If $p_0 \neq 0$ the solution can be represented in the parametric form

$$\Omega = r\left(\frac{2p_0}{z+1}\right)^{1/2}$$

$$\phi = \theta + \sin^{-1}\frac{\Omega_0(z+2)}{(z+1)^{1/2}}$$

$$p = p_0 r^2$$

$$\theta = \phi_0 + \tfrac{1}{2}\sin^{-1}\frac{(1-2\Omega_0)(1+z) - 2\Omega_0^2}{(z+1)(1-4\Omega_0^2)^{1/2}}$$

The solution is the same as ( ), except for the expression for the pressure.

5.2.7. The last subgroup, with operator $X_5 + X_6$, generates the solution $\Omega = e^{\theta}\omega(r)$, $\phi = \theta + \Phi(r)$, and $p = e^{2\theta}P(r)$.

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REFERENCES

Ames, W. F. (1965). *Nonlinear Partial Differential Equations in Engineering*, Academic Press, New York.
Dantzing, L. V. (1934). *Proc. Kon. Akademy, Amsterdam*, **1934**, 35.
Katkov, V. L. (1966). *Izvestiya Akademii Nauk SSSR, Fizika Atmosfery i Okeana*, **II**.
Kucharczyk, P. (1964). *Fluid Dynamics Fractions*, Vol. 83, Pergamon Press, Oxford.
Ladikov, I. P. (1962). *Journal of Applied Mathematics and Mechanics*, **1962**, 26.
Ovsiannpov, L. V. (1962), *Group Properties of Differential Equations*, Novosbirsk.
Singh, S. N., and Choubey, K. R. (1986). *Journal of Mathematical Physics*, No. 5.
Singh, S. N., and Singh, H. P. (1985). *Acta Mechanica* **54**, 191.
Singh, S. N., and Tripathi, D. D. (1986). *Appl. Scientific Research*, 43.
Singh, S. N., and Tripathi, D. D. (1988), *Tamkang Journal of Mathematics*, **16**, to appear.
Surayanarayana, E. R. (1965). *Proceedings of the American Mathematical Society*, **6**, 90.
Yano, K. (1955). *Theory of Lie Derivatives and Its Applications*, North-Holland, Amsterdam.